

SOME STRUCTURAL PROPERTIES OF HOMOMORPHISM DILATION SYSTEMS FOR LINEAR MAPS

DEGUANG HAN, DAVID R. LARSON, BEI LIU, AND RUI LIU

ABSTRACT. Inspired by some recent development on the theory about projection valued dilations for operator valued measures or more generally bounded homomorphism dilations for bounded linear maps on Banach algebras, we explore a pure algebraic version of the dilation theory for linear systems acting on unital algebras and vector spaces. By introducing two natural dilation structures, namely the canonical and the universal dilation systems, we prove that every linearly minimal dilation is equivalent to a reduced homomorphism dilation of the universal dilation, and all the linearly minimal homomorphism dilations can be classified by the associated reduced subspaces contained in the kernel of synthesis operator for the universal dilation.

1. INTRODUCTION

In a recent AMS Memoir [8] we established a general dilation theory for operator valued measures acting on Banach spaces where the operator-valued measures are not necessarily completely bounded. This naturally extends to bounded linear maps acting on Banach algebras and Banach spaces, which can be viewed as a noncommutative analogue for the dilations of operator valued measures. This investigation was mainly motivated by some recent dilation results in frame theory (c.f. [2, 3, 4, 7, 6]), in particular, by a general dilation theorem for framings established by Casazza, Han and Larson [2] which states that every framing (even for a Hilbert space) can have a basis dilation which is highly “non-Hilbertian” in nature and the dilation space has to be a Banach space in general. This is viewed as a true generalization of the well-known Naimark dilation theory [14, 15, 16] for positive operator valued measures, in which case the Hilbertian structure can be completely captured by the dilation space. The Naimark dilation theorem states that every positive operator valued measure has a (self-adjoint) projection valued dilation acting on a Hilbert space. We built in [8, 9, 10] some interesting connections between frame theory and dilations of operator-valued measures on one hand, and the dilations of bounded linear maps between von Neumann algebras on the other hand. It was proved that any operator-valued measure, not necessarily completely bounded, always has a dilation to a projection (idempotent) valued measure acting on a Banach space. More generally, every bounded linear map acting on a Banach algebra has a bounded homomorphism dilation acting on a Banach space. Here the bounded linear map needs not to be completely bounded and the dilation space often needs to be a Banach space even if the underlying space is a Hilbert space, and the underlying algebra is

Key words and phrases. Linear systems, linearly minimal homomorphism dilation systems, principle and universal dilations, equivalent dilation systems .

Deguang Han acknowledges partial support by NSF grants DMS-1106934 and DMS-1403400. Bei Liu and Rui Liu both are supported by NSFC grants 11201336 and 11001134.

a von Neumann algebra. A typical example is the transpose map on the algebra $B(H)$ of all bounded linear operators on a Hilbert space H . This map is not completely bounded but it has a bounded homomorphism dilation on a Banach space and the dilation space can never be taken as a Hilbert space. Such examples also exist for commutative C^* -algebras [8]. Therefore the bounded homomorphism dilation theory for any bounded linear maps truly generalizes the Stinespring's dilation theorem (c.f. [1, 5, 16, 17]) which states that a bounded linear map on a C^* -algebra admits a $*$ -homomorphism dilation (acting on a Hilbert space) if and only if it is completely bounded. It was pointed out in [8] that the problem for the existence of a non-completely bounded linear map that admits a Hilbertian bounded homomorphism dilation is equivalent to Kadison's similarity problem [11]. All these results indicate that it might be possible to establish some kind of classification theory for bounded linear maps based on the properties of their dilations for more general Banach algebras and Banach spaces.

In the dilation theorems for general operator valued measures or general bounded linear maps the dilation Banach space was built on a natural "smallest" dilation vector space equipped with a proper dilation norm so that the involved homomorphisms and linear maps are continuous with respect to the dilation norm. However, neither the (algebraic) dilation space nor the dilation norm is in general not unique. So it seems that there might be some structural theory involved in the "classification" of bounded linear map based on the dilations spaces and the dilation norms, and the completely bounded maps belong to a special class within this structural theory. In order to understand the topological nature of the dilation theory for continuous maps, a good understanding on the purely algebraic aspects of the dilation theory for linear maps is naturally needed. However it seems to us that there is no systematic investigation (at least we are not aware of) in the literature so far. Our aim of this paper is to present several structural results involving the classification of algebraic homomorphism dilations for linear maps acting on general vector spaces. With our ultimate goal of establishing a classification theory of Banach space homomorphism dilations on various dilations spaces, we hope that this paper serves as a first step of this effort.

The rest of the paper is organized as follows: In section 2 we introduce two natural dilations, the canonical dilation and the universal dilation. While the canonical dilation serves as the "smallest" dilation system, the universal one indeed serves as the "largest" dilation system. Naturally we prove that all the irreducible dilations are equivalent to the canonical dilation, and every dilation is equivalent to a reduced dilation of the universal dilation. The main classification results are presented in section 3 in which all the dilations are classified by their associated reduced subspaces contained in the kernel of synthesis operator from the universal dilation. We provide a few remarks and examples in section 4 to demonstrate the complexity and the rich structure of the algebraic dilation theory.

2. PRINCIPLE AND UNIVERSAL DILATIONS

A *linear system* is a triple $(\varphi, \mathcal{A}, V)$ such that φ is a unital linear map from a unital algebra \mathcal{A} to $L(V)$, where V is a vector space and $L(V)$ denotes the space of all linear maps from V to V . In the case that \mathcal{A} is well understood in the discussion we will usually skip \mathcal{A} from the notation.

Definition 2.1. A homomorphism dilation system of a linear system (φ, V) is a unital homomorphism π from \mathcal{A} to a linear operator space $L(W)$ for some vector space W such that there exist an injective linear map $T : V \rightarrow W$ and a surjective linear map $S : W \rightarrow V$ such that for all $a \in \mathcal{A}$ the following diagram commutes

$$\begin{array}{ccc} W & \xrightarrow{\pi(a)} & W \\ T \uparrow & & \downarrow S \\ V & \xrightarrow{\varphi(a)} & V \end{array}$$

That is,

$$\varphi(a) = S\pi(a)T, \quad \forall a \in \mathcal{A}.$$

We will use (π, S, T, W) to denote this homomorphism dilation system, and the dimension of W is called the *dilation dimension* of the homomorphism dilation system (π, S, T, W) . For our convenience we call T as the *analysis operator* and S as the *synthesis operator* for the dilation system. If $\ker(S)$ contains a nonzero π -invariant subspace, then we say that (π, S, T, W) is *reducible*, and otherwise it is called *irreducible*.

Suppose that K is a π -invariant nonzero subspace of $\ker(S)$. Define $\tilde{W} = W/K$, and let $\tilde{S} : \tilde{W} \rightarrow V$, $\tilde{T} : V \rightarrow \tilde{W}$ and $\tilde{\pi} : \mathcal{A} \rightarrow L(\tilde{W})$ be the induced linear maps. Then we have for any $a \in \mathcal{A}$ and any $v \in V$ that

$$\tilde{S}\tilde{\pi}\tilde{T}(v) = \varphi(a)v.$$

Thus $(\tilde{\pi}, \tilde{S}, \tilde{T}, \tilde{W})$ is an homomorphism dilation of (φ, V) and we call it a *reduced homomorphism dilation* of (π, S, T, W) associated with K . If K is the maximal π -invariant subspace contained in $\ker(S)$, then it is easy to show that $\ker(\tilde{S})$ does not contain any nonzero $\tilde{\pi}$ -invariant subspace anymore, and hence the reduced dilation homomorphism system $(\tilde{\pi}, \tilde{S}, \tilde{T}, \tilde{W})$ is irreducible.

Definition 2.2. An homomorphism dilation system (π, W, S, T) of a linear system (φ, V) is called *linearly minimal* if $\text{span}\{\pi(\mathcal{A})TV\} = W$, and it is called a *principle dilation* if it is both linearly minimal and irreducible.

Let (π, S, T, W) be a homomorphism dilation system. Clearly, by replacing W with $\text{span}\{\pi(\mathcal{A})TV\}$, we get a linearly minimal dilation. Then, the reduced dilation system of the new linearly minimal dilation corresponding to the maximal invariant subspace is irreducible. Therefore any homomorphism dilation system leads to a principle dilation system. In what follows we will focused only on linearly minimal dilations.

We construct two very special but important dilations for a given linear system that are essential for our structural theory of dilations. We first introduce the canonical dilation. Let $(\varphi, \mathcal{A}, V)$ be a linear system. For $a \in \mathcal{A}, x \in V$, define $\alpha_{a,x} \in L(\mathcal{A}, V)$ by

$$\alpha_{a,x}(\cdot) := \varphi(\cdot a)x.$$

Let $W := \text{span}\{\alpha_{a,x} : a \in \mathcal{A}, x \in V\} \subset L(\mathcal{A}, V)$. Define $\pi_c : \mathcal{A} \rightarrow L(W)$ by $\pi_c(a)(\alpha_{b,x}) := \alpha_{ab,x}$. It is easy to see that π_c is a unital homomorphism. For $x \in V$ define $T : V \rightarrow L(\mathcal{A}, V)$ by $T_x := \alpha_{I,x} = \phi(\cdot I)x = \phi(\cdot)x$. Define $S : W \rightarrow W$ by setting $S(\alpha_{a,x}) := \phi(a)x$ and extending linearly to W . If $a \in \mathcal{A}, x \in V$ are arbitrary, we have $S\pi_c(a)Tx = S\pi_c(a)\alpha_{I,x} =$

$S\alpha_{a,x} = \phi(a)x$. Hence $\varphi(a) = S\pi_c(a)T$ for all $a \in \mathcal{A}$. Thus (π_c, S, T, W) is a dilation homomorphism of (φ, V) , and we will call it the *canonical dilation* of (φ, V) .

By the construction of W and the definitions of T and π_c it is obvious that (π_c, S, T, W) is a linearly minimal dilation. For the irreducibility, note that

$$\ker(S) = \left\{ \sum_i c_i \alpha_{a_i, x_i} \in W : \sum_i c_i \varphi(a_i) x_i = 0 \right\}$$

Let $w = \sum_i c_i \alpha_{a_i, x_i} \in \ker S$. Then $\pi_c(a)w \in \ker S$ for all $a \in \mathcal{A}$ if and only if $\sum_i c_i \varphi(aa_i)(x_i) = 0$ for all $a \in \mathcal{A}$, which in turn is equivalent to the condition $w = \sum_i c_i \alpha_{a_i, x_i} = 0$ as an element in $L(\mathcal{A}, V)$. Therefore $\ker(S)$ does not contain any nontrivial π_c -invariant subspaces, and consequently we obtain:

Proposition 2.3. *The canonical dilation of a linear system $(\varphi, \mathcal{A}, V)$ is a principle dilation.*

Remark 2.4. In the case that φ is already a unital homomorphism, the canonical dilation π_c must be φ . This can be easily seen by mapping $x \in V$ to $\alpha_{I,x} \in W$. Clearly this is well-defined and linear. The surjectivity follows from the fact that

$$\alpha_{a,x}(b) = \varphi(ba)x = \varphi(b)\varphi(a)x = \alpha_{I,\varphi(a)x}(b)$$

i.e., $\alpha_{a,x} = \alpha_{I,\varphi(a)x}$. With this identification it is easy to see that S and T constructed in the canonical dilation are inverse to each other.

We will see in the next section that the canonical dilation is the one that has the “smallest” dilation dimension and all the principle dilations are equivalent. Note that for any linearly minimal dilation (π, S, T, W) for a finite-dimensional system $(\varphi, \mathcal{A}, V)$, we always have $\dim W \leq (\dim \mathcal{A})(\dim V)$. Now we construct a linearly minimal dilation which has the maximal dilation dimension $(\dim \mathcal{A})(\dim V)$, and we will show later that every linearly minimal dilation system is equivalent to a reduced dilation system of this dilation.

Let $W = \mathcal{A} \otimes V$. Define $\pi_u : \mathcal{A} \rightarrow L(W)$, $S : W \rightarrow V$ and $T : V \rightarrow W$ by the following:

$$\pi_u(a) \left(\sum_i c_i a_i \otimes x_i \right) = \sum_i c_i (ab_i) \otimes x_i,$$

$$Tx = I \otimes x, \quad S \left(\sum_i c_i a_i \otimes x_i \right) = \sum_i c_i \varphi(a_i) x_i.$$

Then π_u is a homomorphism and

$$S\pi_u(a)Tx = S\pi_u(a)(I \otimes x) = S(a \otimes x) = \varphi(a)x$$

for all $x \in V$ and all $a \in \mathcal{A}$. Thus (π_u, S, T, W) is a homomorphism dilation system of (φ, V) . Moreover, since $\pi_u(a)Tx = a \otimes x$, we have $\text{span}\{\pi_u(a)Tx : a \in \mathcal{A}, x \in V\} = W$. Thus (π_u, S, T, W) is a linearly minimal dilation system with the property $\dim W = (\dim \mathcal{A})(\dim V)$.

Definition 2.5. The above constructed dilation (π_u, S, T, W) is called the *universal dilation* of (φ, V) .

3. THE STRUCTURAL THEOREMS

In this section we present our main results about the classifications of all linearly minimal homomorphism dilations.

Definition 3.1. Let (π_1, S_1, T_1, W_1) and (π_2, S_2, T_2, W_2) be two linearly minimal homomorphism dilation systems for a linearly system (φ, V) . We say that the two dilation homomorphism systems are *equivalent* if there exists a bijective linear map $R : W_1 \rightarrow W_2$ such that $RT_1 = T_2$, $S_2R = S_1$ and $\pi_1(a) = R^{-1}\pi_2(a)R$ for all $a \in \mathcal{A}$,

We first point out that $S_2R = S_1$ automatically follows from the other two conditions.

Proposition 3.2. *Let (π_1, S_1, T_1, W_1) and (π_2, S_2, T_2, W_2) be two linearly minimal homomorphism dilation systems for a linearly system (φ, V) . If there exists a bijective linear map $R : W_1 \rightarrow W_2$ such that $RT_1 = T_2$ and $\pi_1(a) = R^{-1}\pi_2(a)R$ for all $a \in \mathcal{A}$, then $S_2R = S_1$ and hence the two systems are equivalent.*

Proof. Since the dilation systems are linearly minimal, we have that $W_j = \text{span } \pi(\mathcal{A})T_j(V)$ for $j = 1, 2$. So for any $w = \sum_i c_i \pi_1(a_i)T_1x_i \in W_1$, we get

$$\begin{aligned} S_2Rw &= S_2\left(\sum_i c_i R\pi_1(a_i)T_1x_i\right) = S_2\left(\sum_i c_i \pi_2(a_i)RT_1x_i\right) \\ &= \sum_i c_i S_2\pi_2(a_i)T_2x_i = \sum_i c_i \varphi(a_i)x_i \\ &= \sum_i c_i S_1\pi_1(a_i)T_1x_i = S_1w. \end{aligned}$$

Thus $S_2R = S_1$. □

The following tells us all the principle homomorphism dilation systems are equivalent:

Theorem 3.3. *If (π_1, S_1, T_1, W_1) and (π_2, S_2, T_2, W_2) are two principle homomorphism dilation systems for $(\varphi, \mathcal{A}, V)$, then (π_1, S_1, T_1, W_1) and (π_2, S_2, T_2, W_2) are equivalent.*

Proof. Since both dilations are linearly minimal, we have $W_i = \text{span } \pi(\mathcal{A})T_i(V)$ for $i = 1, 2$. Define $R : W_1 \rightarrow W_2$ by

$$R(w) = \sum_i c_i \pi_2(a_i)T_2(v_i)$$

if $w = \sum_i c_i \pi_1(a_i)T_1(v_i)$. In order for T to be well-defined and induces the equivalence between π_1 and π_2 , it suffices to show that

$$w = \sum_i c_i \pi_1(a_i)T_1(v_i) = 0$$

if and only if

$$\sum_i c_i \pi_2(a_i)T_2(v_i) = 0.$$

Assume to the contrary that $w \neq 0$. Since

$$S_1 w = \sum_i c_i \varphi(a_i) v_i = S_2 \sum_i c_i \pi_2(a_i) T_2(v_i) = S_2(0) = 0$$

we get that $w \in \ker(S_1)$. Moreover,

$$\begin{aligned} S_1 \pi_1(a) w &= \sum_i c_i S_1 \pi_1(a a_i) T_1(v_i) \\ &= \sum_i c_i S_1 \varphi(a a_i)(v_i) \\ &= \sum_i c_i S_2 \pi_2(a a_i) T_2(v_i) \\ &= S_2 \pi_2(a) \sum_i c_i \pi_2(a_i) T_2(v_i) \\ &= S_2 \pi_2(a)(0) = 0. \end{aligned}$$

Thus, $\pi_1(a)w \in \ker(S_1)$ for all $a \in \mathcal{A}$. So $M = \{\pi_1(a)w : a \in \mathcal{A}\}$ is a nonzero π_1 -invariant subspace inside $\ker(S_1)$, which leads to a contradiction since the dilation (π_1, S_1, T_1, W_1) is irreducible. The argument for the other direction is the same. By the definition of R , we clearly have for any $w = \sum_i c_i \pi_1(a_i) T_1(v_i) \in W_1$ that

$$\begin{aligned} R \pi_1(a) w &= R \sum_i c_i \pi_1(a a_i) T_1(v_i) = \sum_i c_i \pi_2(a a_i) T_2(v_i) \\ &= \pi_2(a) \sum_i c_i \pi_2(a_i) T_2(v_i) = \pi_2(a) R w, \end{aligned}$$

and $R T_1(v) = T_2 v$ for any $v \in V$. Thus we get $\pi_1(a) = R^{-1} \pi_2(a) R$ and $R T_1 = T_2$, and therefore, by Proposition 3.2, we have that (π_1, S_1, T_1, W_1) and (π_2, S_2, T_2, W_2) are equivalent. \square

Corollary 3.4. *Let $(\varphi, \mathcal{A}, V)$ be a linear system such that both \mathcal{A} and V are finite dimensional.*

(i) *Assume that (π, S, T, W) is a principle dilation system of (φ, V) such that $\dim(W) = (\dim \mathcal{A})(\dim(V))$. Then any linearly minimal dilation system of (φ, V) is irreducible, and hence a principle dilation system.*

(ii) *Assume that (π, S, T, W) is a principle dilation system of (φ, V) . If (π_1, S_1, T_1, W_1) is a minimal dilation system of (φ, V) such that $\dim W_1 \leq \dim W$, then it is irreducible.*

Proof. (i) Let (π_1, S_1, T_1, W_1) be a linearly minimal dilation system of (φ, V) . Then

$$\dim W_1 = \dim \text{span}\{\pi_1(\mathcal{A}) T_1 V\} \leq (\dim \mathcal{A})(\dim(V)) = \dim W.$$

Let $(\tilde{\pi}_1, \tilde{S}_1, \tilde{T}_1, \tilde{W}_1)$ be the reduced dilation system of (π_1, S_1, T_1, W_1) corresponding to the maximal π_1 -invariant subspace of $\ker(S_1)$. Then $(\tilde{\pi}_1, \tilde{S}_1, \tilde{T}_1, \tilde{W}_1)$ is both irreducible and linearly minimal. Thus it is a principle dilation system. By Theorem 3.3, we get that π and $\tilde{\pi}_1$ are equivalent, and hence $\dim \tilde{W}_1 = \dim W$. Since $\dim \tilde{W}_1 \leq \dim W_1 \leq \dim W$, we obtain that $\dim W_1 = \dim \tilde{W}_1$, which implies that (π_1, S_1, T_1, W_1) is irreducible.

(ii) Clearly the same argument above also works for part (ii). \square

Corollary 3.5. *Let (π_1, S_1, T_1, W_1) be a linearly minimal dilation system of (φ, V) .*

(i) If $\ker(S_1)$ does not contain any nonzero π -invariant subspaces, then π_1 is equivalent to the canonical homomorphism dilation π_c .

(ii) Assume that $\dim W_1 < \infty$. Then if π_1 is equivalent to the canonical homomorphism π_c , then $\ker(S_1)$ does not contain any nonzero π -invariant subspaces.

Proof. (i) If $\ker(S_1)$ does not contain any nonzero π_1 -invariant subspaces, then by definition it is a principle dilations and hence is equivalent to π_c by Theorem 3.3.

(ii) Assume that π_1 is equivalent to the canonical homomorphism dilation π_c . Then $\dim(W_1) = \dim W$, where W is the dilation space for the canonical dilation. Let K be the largest π -invariant subspace contained in $\ker(S_1)$ and let $(\tilde{\pi}_1, \tilde{S}_1, \tilde{T}_1, W_1/K)$ be the reduced homomorphism dilation system. Then, by Theorem 3.3 again, $\tilde{\pi}_1$ and π_c are equivalent homomorphisms, and so we get $\dim(W) = \dim(W_1/K)$. This implies that $\dim(W_1) = \dim(W_1/K)$. Thus $\dim K = 0$ since $\dim W_1 < \infty$. Therefore $\ker(S_1)$ does not contain any nonzero π -invariant subspaces. \square

Remark 3.6. We don't know if (ii) is still true when $\dim W_1$ is not finite dimensional.

The term of “universal dilation” is justified by the following:

Theorem 3.7. *Any linearly minimal homomorphism dilation of a linear system (φ, V) is equivalent to a reduced homomorphism dilation system of its universal dilation.*

Proof. Let (π_1, S_1, T_1, W_1) be a linearly minimal dilation system. Define K by

$$K = \{w = \sum_i c_i a_i \otimes x_i : \sum_i c_i \pi_1(a_i) T_1 x_i = 0\}.$$

Claim: K is a π_u -invariant subspace contained in $\ker(S)$. In fact, if $w = \sum_i c_i a_i \otimes x_i \in K$, then

$$\begin{aligned} Sw &= \sum_i c_i S(a_i \otimes x_i) = \sum_i c_i S \pi_u(a_i) T x \\ &= \sum_i c_i \varphi(a_i) x_i = \sum_i c_i S_1 \pi_1(a_i) T_1 x_i \\ &= S_1 \sum_i c_i \pi_1(a_i) T_1 x_i = S(0) = 0. \end{aligned}$$

Thus $K \subseteq \ker(S)$. Moreover, for any $a \in \mathcal{A}$ and $w = \sum_i c_i a_i \otimes x_i \in K$, we have

$$\sum_i c_i \pi_1(aa_i) T_1 x_i = \pi_1(a) \sum_i c_i \pi_1(a_i) T_1 x_i = 0.$$

Thus $\pi_u(a)w = \sum_i c_i (aa_i) \otimes x_i \in K$. Therefore K is a π_u -invariant subspace contained in $\ker(S)$.

Let $(\tilde{\pi}_u, \tilde{S}, \tilde{T}, W/K)$ be the reduced dilation homomorphism. Define $R : W/K \rightarrow W_1$ by

$$R[w] = \sum_i c_i \pi_1(a_i) T_1 x_i$$

for any $[w] \in W/K$ represented by $w = \sum_i c_i a_i \otimes x_i$. Then, by the definition of K , we have $R[w] = \sum_i c_i a_i \otimes x_i = 0$ if and only if $w \in K$. Hence, R is a well defined injective

linear map. Clearly it is also surjective since $\text{span}\{\pi_1(\mathcal{A})T_1V\} = W_1$. Moreover, for any $w = \sum_i c_i a_i \otimes x_i \in W$, we have

$$\begin{aligned} \pi_1(a)R([w]) &= \pi_1(a) \sum_i c_i \pi_1(a_i) T_1 x_i = \sum_i c_i \pi_1(aa_i) T_1 x_i \\ &= R\left(\sum_i c_i (aa_i) \otimes x_i\right) = R\tilde{\pi}_u(a)([w]) \end{aligned}$$

Thus $\pi_1(a) = R\tilde{\pi}_u(a)R^{-1}$ for any $a \in \mathcal{A}$. Moreover, for any $w = \sum_i c_i a_i \otimes x_i \in W$ we have

$$R\tilde{T}x = R[Tx] = R[I \otimes x] = \pi_1(I)T_1x.$$

Hence $R\tilde{T} = T_1$. Therefore (π_1, S_1, T_1, W_1) and $(\tilde{\pi}_u, \tilde{S}, \tilde{T}, W/K)$ are equivalent. \square

In order to classify the linearly minimal homomorphism dilation systems we introduce the following:

Definition 3.8. Let (π_u, S, T, W) be the universal dilation system and (π_1, S_1, T_1, W_1) be a linearly minimal homomorphism dilation system for a linear system (φ, V) . Then the π_u -invariant subspace K_1 introduced in the above proof will be called the *reduced subspace* associated with (π_1, S_1, T_1, W_1) .

Remark 3.9. We point out that the reduced subspace K of a linearly minimal homomorphism dilation system (π_1, S_1, T_1, W_1) is different from the maximal π_1 -invariant subspace M contained in $\ker(S_1)$ which is used to reduce (π_1, S_1, T_1, W_1) to the “smallest” dilation — the principle dilation, while K is a π_u -invariant subspace contained in the universal dilation space W (i.e., $\mathcal{A} \otimes V$) that is used to reduce the universal dilation system to (π_1, S_1, T_1, W_1) .

The following gives us a classification of all linearly minimal homomorphism dilations systems for a given linear system.

Theorem 3.10. Let K_1 and K_2 be the reduced subspaces associated with the linearly minimal homomorphism dilation systems (π_1, S_1, T_1, W_1) and (π_2, S_2, T_2, W_2) , respectively. Then the two homomorphism dilation systems (π_1, S_1, T_1, W_1) and (π_2, S_2, T_2, W_2) are equivalent if and only if $K_1 = K_2$.

Proof. By Theorem 3.7 we only need to prove that if (π_1, S_1, T_1, W_1) and (π_2, S_2, T_2, W_2) are equivalent, then $K_1 = K_2$. Let $R : W_1 \rightarrow W_2$ be a bijective linear map such that $\pi_2(a)R = R\pi_1(a)$ for all $a \in \mathcal{A}$, $S_2R = S_1$ and $RT_1 = T_2$.

Let $w = \sum_i c_i a_i \otimes x_i \in W$. Since

$$\sum_i c_i \pi_1(a_i) T_1 x_i = R^{-1} \sum_i c_i \pi_2(a_i) R T_1 x_i = R^{-1} \sum_i c_i \pi_2(a_i) T_2 x_i,$$

we get that $\sum_i c_i \pi_1(a_i) x_i = 0$ if and only if $\sum_i c_i \pi_2(a_i) T_2 x_i = 0$, i.e., $w \in K_1$ if and only if $w \in K_2$. Hence $K_1 = K_2$. \square

The above theorem shows that the equivalent class of linearly minimal homomorphism dilation systems is uniquely determined by the reduced subspace. We will show by example in section 4 that there could be infinitely many inequivalent linearly minimal homomorphism dilation systems even in the finite-dimensional case (i.e., $\dim V < \infty$ and $\dim(\mathcal{A}) < \infty$).

Additionally, there is a weaker version of equivalence which seems also relevant to the dilation theory: If (π_1, S_1, T_1, W_1) be a linearly minimal dilation system for a linear system (φ, V) , and π_2 is a homomorphism from \mathcal{A} to $L(W_2)$ such that π_1 and π_2 are equivalent in the usual sense, i.e. $\pi_1(a) = R^{-1}\pi_2(a)R$ ($\forall a \in \mathcal{A}$) for some isomorphism $R : W_1 \rightarrow W_2$, then (π_2, S_2, T_2, W_2) is an equivalent dilation system with $S_2 = S_1R^{-1}$ and $T_2 = RT_1$. Thus it is interesting to know that under what condition do we have two equivalent homomorphisms π_1 and π_2 for linearly minimal homomorphism dilation systems (π_1, S_1, T_1, W_1) and (π_2, S_2, T_2, W_2) . For this purpose we introduce the following concept of equivalence for the reducing invariant subspaces.

Definition 3.11. Let (π_u, S, T, W) be the universal dilation system of a linearly system (φ, V) . Two π_u -invariant subspaces K_1 and K_2 of $\ker(S)$ are called *strongly isomorphic* if there is an isomorphism $R : W \rightarrow W$ such that $R(K_1) = K_2$ and $\pi_u(a)Rw - R\pi_u(a)w \in K_2$ for all $a \in \mathcal{A}$ and all $w \in W$, i.e., the quotient maps of $\pi_u(a)$ and R on W/K_2 commute for all $a \in \mathcal{A}$.

Theorem 3.12. Let K_1 and K_2 be the reduced subspaces for the linearly minimal homomorphism dilation systems (π_1, S_1, T_1, W_1) and (π_2, S_2, T_2, W_2) , respectively. Then π_1 and π_2 are equivalent if and only if K_1 and K_2 are strongly isomorphic.

Proof. By Theorem 3.7 we can assume that (π_i, S_i, T_i, W_i) is the reduced homomorphism dilation of the universal dilation associated with K_i ($i = 1, 2$).

(\Leftarrow): Assume that K_1 and K_2 are strongly isomorphic. Then there is an isomorphism $R : W \rightarrow W$ such that $R(K_1) = K_2$ and $\pi_u(a)Rw - R\pi_u(a)w \in K_2$ for all $a \in \mathcal{A}$ and all $w \in W$. Let $\tilde{R} : W_1 = W/K_1 \rightarrow W/K_2 = W_2$ be defined by

$$\tilde{R}[w] = [Rw], \quad w \in W,$$

where we use $[\cdot]$ to denote the element in the corresponding quotient space. Then \tilde{R} is a bijective linear transformation. Note that since π_2 is the reduced homomorphism of π_u on W/K_2 , we have that $\pi_2(a)\tilde{R}([w]) = \pi_2(a)[Rw] = [\pi_u(a)Rw]$. Similarly, $\tilde{R}\pi_1(a)[w] = \tilde{R}[\pi_u(a)w] = [R\pi_u(a)w]$. Thus, from $\pi_u(a)Rw - R\pi_u(a)w \in K_2$, we obtain that $\pi_2(a)\tilde{R}[w] = \tilde{R}\pi_1(a)[w]$, which implies that π_1 and π_2 are equivalent.

(\Rightarrow): Assume that π_1 and π_2 are equivalent. Then there is bijective linear map $L : W/K_1 \rightarrow W/K_2$ such that $\pi_2(a)L = L\pi_1(a)$ for all $a \in \mathcal{A}$. Since $\dim(K_1) = \dim(K_2)$, we obtain that there exists a bijective linear map $R : W \rightarrow W$ such that the $R(K_1) = K_2$ and the induced quotient map \tilde{R} is L . Moreover, from $\pi_2(a)L = L\pi_1(a)$ we have that $\pi_2(a)\tilde{R} = \tilde{R}\pi_1(a)$, which is equivalent to the condition that $\pi_u(a)Rw - R\pi_u(a)w \in K_2$ for all $a \in \mathcal{A}$ and all $w \in W$. Thus K_1 and K_2 are strongly isomorphic. \square

4. REMARKS AND EXAMPLES

Theorem 3.10 and Theorem 3.12 provide us with two classifications for linearly minimal homomorphism dilations based on the universal dilation invariant subspaces in the kernel of the map $S : \mathcal{A} \otimes V \rightarrow V$ defined by $S(a \otimes x) = \varphi(a)x$. These lead to many interesting questions, especially in the finite dimensional case. For example, (1) under what condition on $(\varphi, \mathcal{A}, V)$ do we have the property that for every k between the dimensions of V and $\mathcal{A} \otimes V$ there exists a linearly minimal dilation with dilation dimension k . (2) When do we

have only finite many inequivalent linearly minimal homomorphism dilations? (Examples 4.5 and 4.7 show that we could have infinitely many inequivalent classes even both \mathcal{A} and V are finite dimensional.) (3) Under what condition do we have that the principle and universal dilations are the only two classes of linearly minimal dilations? (4) We will construct an example showing that there exist reduced subspaces K_1 and K_2 that are strongly isomorphic by $K_1 \neq K_2$. However, it would be interesting to know that if the condition $\dim K_1 = \dim K_2$ automatically implies that they are strongly isomorphic.

In what follows we will answer some of these questions and at the same time constructing some examples showing the complexity of other questions.

Let $M = \{\sum_i c_i a_i x_i : \sum_i c_i \varphi(aa_i)x_i = 0, \forall a \in \mathcal{A}\}$. Then M is the largest π_u -invariant subspace contained in $\ker(S)$. Hence, by Theorem 3.3 we have that the universal homomorphism dilation equivalent to the principle dilation if and only if $M = \{0\}$. Moreover we have

Proposition 4.1. *A linear system (φ, V) has only one equivalent class of linearly minimal homomorphism dilations if and only if $M = \{0\}$.*

Proof. Let (π_1, S_1, T_1, W_1) be a linearly minimal dilation homomorphism system for (φ, V) . Let K_1 be its reduced subspace. If $w = \sum_i c_i a_i x_i \in K_1$, then $\sum_i c_i \phi_1(aa_i)T_1 x_i = 0$ for every $a \in \mathcal{A}$. Since $\varphi(\cdot) = S_1 \pi_1(\cdot) T_1$, we get that $\sum_i c_i \varphi(aa_i)x_i = 0$ for all $a \in \mathcal{A}$, i.e., $w \in M$. Thus $K_1 = \{0\}$, and so (π_1, S_1, T_1, W_1) is equivalent to the universal dilation. \square

Corollary 4.2. *Let $(\varphi, \mathcal{A}, V)$ be a linear system. If $\ker(\varphi)$ contains a proper left ideal, then the universal dilation is not equivalent to its principle dilation.*

Proof. Let a be a nonzero element in the left ideal. Then for any $x \in V$ and any $b \in \mathcal{A}$ we have $\varphi(ba)x = 0$, which implies that $a \otimes x \in M$. Hence $M \neq \{0\}$ and consequently the universal dilation is not equivalent to the principle dilation. \square

Note that if $\dim(V) = 1$, then $\mathcal{A} \otimes V = \{a \otimes x : a \in \mathcal{A}\}$, where x is a fixed nonzero vector in V . So $M = \{a \otimes x : \varphi(ba)x = 0, \forall b \in \mathcal{A}\} = \{a \otimes x : \varphi(ba) = 0, \forall b \in \mathcal{A}\}$, where we used the factor that $\varphi(ba)$ is a scalar. Thus we get

Corollary 4.3. *Let $(\varphi, \mathcal{A}, V)$ be a linear system such that $\dim(V) = 1$. Then its universal dilation and principle dilation are equivalent if and only if $\ker(\varphi)$ does not contain any proper left ideals.*

Example 4.4. Let $\mathcal{A} = \mathbb{M}_n$ be the $n \times n$ matrix algebra, and $\varphi(A) = \frac{1}{n} \text{tr}(A)$. Then it is easy to show that $\ker(\varphi)$ does not contain any proper left ideals, and hence the universal dilation is the same as its canonical dilation. For example if $n = 2$, then $\varphi(A) = \frac{a+d}{2}$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then the canonical (as well as the universal) homomorphism dilation system $(\pi, S, T, \mathbb{C}^4)$ is given by

$$\pi(A) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$$

with

$$S = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}^t.$$

Example 4.5. Let $\mathcal{A} = \mathcal{T}_n$ be the algebra of all the $n \times n$ upper triangular matrices, $\mathcal{T}_{n,0}$ be the algebra of all the $n \times n$ strictly upper triangular matrices, and $\varphi(a) = \frac{1}{n} \text{tr}(a)$. Then $\mathcal{T}_{n,0}$ is a proper ideal contained in $\ker(\varphi)$. Thus the universal dilation system is not equivalent to its canonical dilation system.

(i) For $n = 2$ we have $\varphi(A) = \frac{1}{2}(a + c)$ where

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

Then the universal dilation system $(\pi_u, S_u, T_u, \mathbb{C}^3)$ and the canonical dilation system $(\pi_c, S_c, T_c, \mathbb{C}^2)$ are given by:

$$\pi_u(A) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & c \end{pmatrix}, \quad \text{and} \quad \pi_c(A) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix},$$

where

$$S_u = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \quad T_u = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^t,$$

and

$$S_c = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad T_c = \begin{pmatrix} 1 & 1 \end{pmatrix}^t.$$

These are the only two linearly minimal homomorphism dilations.

(ii) For the case $n = 3$, we have $\varphi(A) = \frac{1}{3}(a + d + f)$ where

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

In this case the universal dilation system $(\pi_u, S_u, T_u, \mathbb{C}^6)$ and the canonical dilation system $(\pi_c, S_c, T_c, \mathbb{C}^3)$ are given by:

$$\pi_u(A) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & 0 & 0 & f \end{pmatrix}, \quad \text{and} \quad \pi_c(A) = \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{pmatrix},$$

where

$$S_u = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad T_u = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}^t,$$

and

$$S_c = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad T_c = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^t.$$

In order to identify the rest of the equivalent classes of homomorphism dilations, we need to identify all the π_u -invariant subspaces in $\ker(S_u)$. Note that $\ker(S_u) = \text{span}\{e_2, e_4, e_5\}$, and it is easy to verify that the maximal π_u -invariant subspace is $\text{span}\{e_2, e_4\}$, and any one-dimensional subspace of $\text{span}\{e_2, e_4\}$ is also π_u -invariant. Hence, by Theorem 3.10,

we only have one equivalent class of 4-dimensional homomorphism dilation systems, and infinitely many inequivalent class of 5-dimensional homomorphism dilation systems.

The 4-dimensional equivalent class of homomorphism dilation systems is represented by $(\pi_4, S_4, T_4, \mathbb{C}^4)$:

$$S_4 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}, \quad \pi_4(A) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix}^t.$$

Two classes of homomorphism dilations systems represented by $(\pi_{5,1}, S_{5,1}, T_{5,1}, \mathbb{C}^5)$ and $(\pi_{5,2}, S_{5,2}, T_{5,2}, \mathbb{C}^5)$ that associated with π_u -invariant subspaces $K_1 = \text{span}\{e_2\}$ and $K_2 = \text{span}\{e_4\}$, respectively, are given by:

$$S_{5,1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad \pi_{5,1}(A) = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 \\ 0 & 0 & a & b & c \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & 0 & f \end{pmatrix}, \quad T_5 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \end{pmatrix}^t$$

and

$$S_{5,2} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}, \quad \pi_{5,2}(A) = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & 0 & f \end{pmatrix}, \quad T_{5,2} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \end{pmatrix}^t$$

We leave the construction of the homomorphism dilation associated with the π_u -invariant subspace $K_{\alpha,\beta} = \text{span}\{\alpha e_2 + \beta e_4\}$ for the interested readers.

Example 4.6. Let $\varphi : \mathbb{M}_2 \rightarrow \mathbb{M}_2$ be defined by

$$\varphi \left(\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) = \begin{pmatrix} \sum_{i=1}^4 a_i \alpha_i & \sum_{i=1}^4 b_i \alpha_i \\ \sum_{i=1}^4 c_i \alpha_i & \sum_{i=1}^4 d_i \alpha_i \end{pmatrix}.$$

Then we have the universal dilation $\pi_u : \mathbb{M}_2 \rightarrow \mathbb{M}_8$ given by

$$\pi_u \left(\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_3 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & \alpha_4 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3 & \alpha_4 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_3 & \alpha_4 \end{pmatrix}$$

with

$$S_u = \begin{pmatrix} a_1 & a_3 & a_2 & a_4 & b_1 & b_3 & b_2 & b_4 \\ c_1 & c_3 & c_2 & c_4 & d_1 & d_3 & d_2 & d_4 \end{pmatrix}$$

and

$$T_u = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Let τ and σ the linear maps on \mathbb{M}_2 defined by $\tau(A) = A^t$ and $\sigma(A) = \frac{1}{2}Tr(A)I$, where $I \in \mathbb{M}_2$ is the identity matrix. Then it can be shown that there is no nontrivial π_u -invariant subspaces in $\ker(S_u)$, and thus the above formula also gives us the canonical dilation. The situation becomes quite different for transpose map of triangular matrices. For simplicity let us examine the transpose map on \mathcal{T}_2 and \mathcal{T}_3 .

Example 4.7. Let $\tau : \mathcal{T}_2 \rightarrow \mathbb{M}_2$ be the transpose map. Then the universal dilation system is given by:

$$\pi_u \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}$$

with

$$S_u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_u = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}^t.$$

The canonical dilation system is given by:

$$\pi_c \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & c \end{pmatrix}$$

with

$$S_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad T_c = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}^t.$$

Furthermore, we have

$$\ker S_u = \text{span}\{e_2 - e_6, e_3, e_4, e_5\}.$$

In $\ker S_u$, the maximal π_u -invariant subspace is $M = \text{span}\{e_4, e_5\}$, and for any given α, β , the one-dimensional subspace $K_{\alpha, \beta} = \text{span}\{\alpha e_4 + \beta e_5\}$ is π_u -invariant. So, again, there are infinitely many inequivalent classes of 5-dimensional dilations. The two special ones corresponding to $K_{1,0}$ and $K_{0,1}$ are represented by:

The

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

(ii) Let $\tau : \mathcal{T}_3 \rightarrow \mathbb{M}_3$ be the transpose map

$$\tau \left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \right) = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}.$$

Then we have the canonical dilation $\pi_c : \mathcal{T}_3 \rightarrow \mathbb{M}_{10}$ by

$$\pi_c \left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \right) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 & 0 & d & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f \end{pmatrix}$$

with

$$S_c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Example 4.8. Let $v : \mathbb{M}_2 \rightarrow \mathbb{M}_2$ be the linear map

$$v \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \alpha_1(\xi_1 a + \xi_2 b) + \alpha_2(\xi_1 c + \xi_2 d) & 0 \\ 0 & \beta_1(\gamma_1 a + \gamma_2 b) + \beta_2(\gamma_1 c + \gamma_2 d) \end{pmatrix}.$$

Then we have the (linearly minimal) dilation $\pi : \mathcal{T}_2 \rightarrow \mathbb{M}_4$ by

$$\pi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$$

with

$$S = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} \xi_1 & 0 \\ \xi_2 & 0 \\ 0 & \gamma_1 \\ 0 & \gamma_2 \end{pmatrix}.$$

We remark that this is the principle dilation for the following maps d and ϕ on M_2 :

$$d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad \text{and} \quad \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

REFERENCES

- [1] W. Arveson, *Dilation theory yesterday and today*, A glimpse at Hilbert space operators, 99–123, Oper. Theory Adv. Appl., 207, Birkhäuser Verlag, Basel, 2010.
- [2] P. G. Casazza, D. Han, and D. R. Larson, *Frames for Banach spaces*, The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), Contemp. Math. **247** (1999), 149–182.
- [3] J.-P. Gabardo and D. Han, *Frames associated with measurable spaces*, Adv. Comput. Math. **18** (2003) 127–147.
- [4] J.-P. Gabardo and D. Han, *Frame representations for group-like unitary operator systems*, J. Operator Theory, **49** (2003), 223–244.
- [5] D. W. Hadwin, *Dilations and Hahn decompositions for linear maps*, Canad. J. Math. **33** (1981), 826–839.
- [6] D. Han, *Dilations and completions for Gabor systems*, J. Fourier Anal. Appl., **15** (2009), 201–217.
- [7] D. Han and D. R. Larson, *Frames, bases and group representations*, Mem. Amer. Math. Soc. **697** (2000), 1–94.
- [8] D. Han, D.R. Larson, B. Liu and R. Liu, *Operator-Valued Measures, Dilations, and the Theory of Frames*, Mem. Amer. Math. Soc., Vol. 229, No. 1075 (2014).
- [9] D. Han, D.R. Larson, B. Liu and R. Liu, *Dilations for systems of imprimitivity acting on Banach spaces*, J. Funct. Anal., **266** (2014), 6914–6937.
- [10] D. Han, D.R. Larson, B. Liu and R. Liu, *Dilations of frames, operator valued measures and bounded linear maps*, Contemp. Math., Amer. Math. Soc. ,
- [11] R. Kadison, *On the orthogonalization of operator representations*, Amer. J. Math., **77**(1955), 600–622.
- [12] V. Kaftal, D. R. Larson, and S. Zhang, *Operator valued frames*, Trans. Amer. Math. Soc. **361** (2009), 6349–6385.
- [13] G. W. Mackey, *Imprimitivity for representations of locally compact groups*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 537–545.
- [14] M. A. Naimark, *Spectral functions of a symmetric operator*, Izv. Akad. Nauk SSSR Ser. Mat., **4**(1940), no. 3, pp. 277–318.
- [15] M. A. Naimark, *On a representation of additive operator set functions*, Dokl. Acad. Sci. SSSR, **41**(1943), no. 9, pp. 373–375.
- [16] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge University Press (2002).
- [17] W. F. Stinespring, *Positive Functions on C^* -algebras*, Proc. Amer. Math. Soc., **6** (1955), 211–216.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, USA
E-mail address: deguang.han@ucf.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, USA
E-mail address: larsen@math.tamu.edu

DEPARTMENT OF MATHEMATICS, TIANJIN UNIVERSITY OF TECHNOLOGY, TIANJIN, CHINA
E-mail address: beiliu1101@gmail.com

DEPARTMENT OF MATHEMATICS AND LPMC, NANKAI UNIVERSITY, TIANJIN, CHINA

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, USA

E-mail address: `ruiliu@nankai.edu.cn`; `rliu@math.tamu.edu`